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NONABELIAN GROUP ACTIONS ON 3-DIMENSIONAL NILMANIFOLDS REVERSING FIBER ORIENTATION

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Abstract. We study free actions of finite nonabelian groups on 3-dimensional nilmanifolds with the first homology $\mathbb{Z}^2 \oplus \mathbb{Z}_2$ which yield an orbit manifold reversing fiber orientation, up to topological conjugacy. We show that those nonabelian groups are D_4(the dihedral group), Q_8(the quaternion group), and $C_8.C_4$(the 1^{st} non-split extension by C_8 of C_4 acting via $C_4/C_2 = C_2$).

1. Introduction

Let $\mathcal{H}$ be the 3–dimensional Heisenberg group; i.e. $\mathcal{H}$ consists of all 3×3 real upper triangular matrices with diagonal entries 1. Thus $\mathcal{H}$ is a simply connected, 2-step nilpotent Lie group, and it fits an exact sequence

$$1 \to \mathbb{R} \to \mathcal{H} \to \mathbb{R}^2 \to 1$$

where $\mathbb{R} = \mathcal{Z}(\mathcal{H})$, the center of $\mathcal{H}$. Hence $\mathcal{H}$ has the structure of a line bundle over $\mathbb{R}^2$. We take a left invariant metric coming from the orthonormal basis

$$\left\{ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

for the Lie algebra of $\mathcal{H}$. This is, what is called, the Nil-geometry and its isometry group is $\mathrm{Isom}(\mathcal{H}) = \mathcal{H} \rtimes O(2)$ [8]. All isometries of $\mathcal{H}$ preserve orientation and the bundle structure.

We say that a closed 3-dimensional manifold M has a Nil-geometry if there is a subgroup π of $\mathrm{Isom}(\mathcal{H})$ so that π acts properly discontinuously

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and freely on $\mathcal{H}$ with quotient $M = \mathcal{H}/\pi$. The simplest such a manifold is the quotient of $\mathcal{H}$ by the lattice consisting of integral matrices.

Let Γ be any lattice of $\mathcal{H}$ and $\mathcal{Z}(\mathcal{H})$ be the center of $\mathcal{H}$. Then $\mathbb{Z} = \Gamma \cap \mathcal{Z}(\mathcal{H})$ and $\Gamma/\Gamma \cap \mathcal{Z}(\mathcal{H})$ are lattices of $\mathcal{Z}(\mathcal{H})$ and $\mathcal{H}/\mathcal{Z}(\mathcal{H})$, respectively. Therefore, the lattice Γ is an extension of $\mathbb{Z}$ by $\mathbb{Z}^2$, that is, there is an exact sequence:

$$1 \to \mathbb{Z} \to \Gamma \to \mathbb{Z}^2 \to 1.$$

Let a, b, and c be elements of Γ such that the images of a and b in $\mathbb{Z}^2$ generate $\mathbb{Z}^2$ and c generates the center $\mathbb{Z}$. Then it is known that such Γ is isomorphic to one of the following groups, for some p :

$$\Gamma_p = \langle a, b, c \mid [b, a] = c^p,\ [c, a] = [c, b] = 1 \rangle, \quad p \neq 0,$$

where $[b, a] = b^{-1}a^{-1}ba$. This group is realized as a uniform lattice of $\mathcal{H}$ if one takes

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},\ b = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ c = \begin{bmatrix} 1 & 0 & \frac{1}{p} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then Γ_1 is the discrete subgroup of $\mathcal{H}$ consisting of all integral matrices and Γ_p is a lattice of $\mathcal{H}$ containing Γ_1 with index p. Remark that Γ_p is equal to Γ_{-p}. Clearly

$$\mathrm{H}_1(\mathcal{H}/\Gamma_p; \mathbb{Z}) = \Gamma_p/[\Gamma_p, \Gamma_p] = \mathbb{Z}^2 \oplus \mathbb{Z}_p.$$

Note that these ${\Gamma_p}'$s produce infinitely many distinct nilmanifolds

$$\mathcal{N}_p = \mathcal{H}/\Gamma_p$$

covered by the standard nilmanifold $\mathcal{N}_1$.

In [2], the authors showed that if a finite group acts freely on the standard 3-dimensional nilmanifold $\mathcal{N}_1$ with the first homology $\mathbb{Z}^2$, then it is cyclic. Also, they showed that there does not exist any finite group acting freely (up to topological conjugacy) on $\mathcal{N}_1$ which yields an orbit manifold homeomorphic to $\mathcal{H}/\pi_3$ or $\mathcal{H}/\pi_4$. Later, free actions of finite abelian groups on the 3-dimensional nilmanifold $\mathcal{N}_p$ with the first homology $\mathbb{Z}^2 \oplus \mathbb{Z}_p$ were classified in [1]. In [5], the authors generalized the results of [1] by changing the finite abelian group conditions to finite group conditions by using the method in [1], up to topological conjugacy. However it is difficult to know exactly what the finite groups are, since the finite groups acting freely on $\mathcal{N}_p$ are represented by generators in [5].

In this paper we study free actions of finite *nonabelian* groups on $\mathcal{N}_2$ yielding orbit manifolds reversing fiber orientation, which is homeomorphic to $\mathcal{H}/\pi_3$ or $\mathcal{H}/\pi_4$. We will prove in Theorem 3.3 and Theorem 3.6 that the nonabelian groups are isomorphic to D_4(the dihedral group), Q_8(the quaternion group), or $C_8.C_4$(the 1^{st} non-split extension by C_8 of C_4 acting via $C_4/C_2 = C_2$).

Note that our results cannot be obtained directly from [5]. But when $p = 2$, we can find a necessary and sufficient condition for being a normal nilpotent subgroup of an almost Bieberbach group, and classify exactly what those groups are. This classification problem is reduced to classifying all normal nilpotent subgroups of almost Bieberbach groups of finite index, up to affine conjugacy.

2. Criteria for affine conjugacy

In this section, we develop a technique for finding and classifying all possible finite group actions on 3-dimensional nilmanifolds with the first homology $\mathbb{Z}^2 \oplus \mathbb{Z}_p$. The problem will be reduced to a purely group-theoretic one. We quote most of the Introduction and Section 2 of [1] in this section for the reader's convenience.

Let G be a finite group acting freely on the nilmanifold $\mathcal{N}_p$. Then clearly, $M = \mathcal{N}_p/G$ is a topological manifold, and $\pi = \pi_1(M) \subset \mathrm{TOP}(\mathcal{H})$ is isomorphic to an almost Bieberbach group. Let π' be an embedding of π into $\mathrm{Aff}(\mathcal{H})$. Such an embedding always exists. Since any isomorphism between lattices extends uniquely to an automorphism of $\mathcal{H}$, we may assume the subgroup Γ_p goes to itself by the embedding $\pi \to \pi' \subset \mathrm{Aff}(\mathcal{H})$. Then the quotient group $G' = \pi'/\Gamma_p$ acts freely on the nilmanifold $\mathcal{N}_p = \mathcal{H}/\Gamma_p$. Moreover, $M' = \mathcal{N}_p/G'$ is an infra-nilmanifold. Thus, a finite free topological action $(G, \mathcal{N}_p)$ gives rise to an isometric action $(G', \mathcal{N}_p)$ on the nilmanifold $\mathcal{N}_p$. By works of Waldhausen and Heil ([4, 9]), M is homeomorphic to M'.

Definition 2.1. Let groups G_i act on manifolds M_i, for $i = 1, 2$. The action (G_1, M_1) is *topologically conjugate* to (G_2, M_2) if there exists an isomorphism $\theta : G_1 \to G_2$ and a homeomorphism $h : M_1 \to M_2$ such that

$$h(g \cdot x) = \theta(g) \cdot h(x)$$

for all $x \in M_1$ and all $g \in G_1$. When $G_1 = G_2$ and $M_1 = M_2$, topologically conjugate is the same as *weakly equivariant*.

For $\mathcal{N}_p/G$ and $\mathcal{N}_p/G'$ being homeomorphic implies that the two actions $(G, \mathcal{N}_p)$ and $(G', \mathcal{N}_p)$ are topologically conjugate. Such a pair $(G', \mathcal{N}_p)$ is not unique. However, by the result obtained by Lee and Raymond [7], all the others are topologically conjugate.

The following proposition gives a characterization of an almost Bieberbach group (see [6]).

Proposition 2.2. *An abstract group π is the fundamental group of a 3-dimensional infra-nilmanifold if and only if π is torsion-free and contains Γ_k for some $k > 0$ as a maximal normal nilpotent subgroup of finite index.*

It is known ([3, Proposition 6.1]) that there are 15 classes of distinct closed 3-dimensional manifolds M with a Nil-geometry.

Note that if $M = \mathcal{H}/\pi$ is a 3-dimensional infra-nilmanifold, then there is a diffeomorphism f between $\mathcal{H}$ and $\mathbb{R}^3$, and an isomorphism φ between π and π', where π' is a subgroup of

$$\mathrm{Aff}(\mathbb{R}^3) = \mathbb{R}^3 \rtimes \mathrm{GL}(3, \mathbb{R})$$

such that $(\pi, \mathcal{H})$ and $(\pi', \mathbb{R}^3)$ are weakly equivariant. Therefore, an infra-nilmanifold $M = \mathcal{H}/\pi$ is diffeomorphic to an affine manifold $M' = \mathbb{R}^3/\pi'$.

The list for 15 kinds of the 3-dimensional almost Bieberbach groups imbedded in $\mathrm{Aff}(\mathcal{H}) = \mathcal{H} \rtimes (\mathbb{R}^2 \rtimes \mathrm{GL}(2, \mathbb{R}))$ is presented in [1, p.799-p.801]. We shall use

$$t_1 = \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I \right), t_2 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, I \right), t_3 = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I \right),$$

respectively, where I is the identity in $\mathrm{Aut}(\mathcal{H}) = \mathbb{R}^2 \rtimes \mathrm{GL}(2, \mathbb{R})$.

Let $(G, \mathcal{N}_p)$ be a free affine action of a finite group G on the nilmanifold $\mathcal{N}_p$. Then $\mathcal{N}_p/G$ is an infra-nilmanifold. Let $\pi = \pi_1(\mathcal{N}_p/G)$ and $\Gamma_p = \pi_1(\mathcal{N}_p)$. Then π is an almost Bieberbach group. In fact, since the covering projection $\mathcal{N}_p \to \mathcal{N}_p/G$ is regular, Γ_p is a normal subgroup of π.

Definition 2.3. Let $\pi \subset \mathrm{Aff}(\mathcal{H}) = \mathcal{H} \rtimes \mathrm{Aut}(\mathcal{H})$ be an almost Bieberbach group, and let N_1, N_2 be subgroups of π. We say that (N_1, π) is *affinely conjugate* to (N_2, π), denoted by $N_1 \sim N_2$, if there exists an element $(t, T) \in \mathrm{Aff}(\mathcal{H})$ such that $(t, T)\pi(t, T)^{-1} = \pi$ and $(t, T)N_1(t, T)^{-1} = N_2$.

Our classification problem of free finite group actions $(G, \mathcal{N}_p)$ with

$$\pi_1(\mathcal{N}_p/G) \cong \pi$$

can be solved by finding all normal nilpotent subgroups N of π each of which is isomorphic to Γ_p, and classify (N, π) up to affine conjugacy. This procedure is a purely group-theoretic problem and can be handled by affine conjugacy.

The following proposition [1, Proposition 3.1] is a working criterion for determining all normal nilpotent subgroups of π which are isomorphic to Γ_p.

Proposition 2.4. *Let N be a normal nilpotent subgroup of an almost Bieberbach group π and isomorphic to Γ_p. Then N can be represented by a set of generators*

$$N = \langle\, t_1^{d_1} t_2^{m} t_3^{n_1},\, t_2^{d_2} t_3^{n_2},\, t_3^{\frac{Kd_1d_2}{p}} \,\rangle,$$

where d_1, d_2 are divisors of p; K is determined by $t_3^K = [t_2, t_1]$; $0 \le m < d_2$, $0 \le n_i < \frac{Kd_1d_2}{p}$ $(i = 1, 2)$.

3. Free actions of finite nonabelian groups on the nilmanifold $\mathcal{N}_2$

We shall find all possible finite nonabelian groups acting freely (up to topological conjugacy) on the 3-dimensional nilmanifold $\mathcal{N}_2$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_3$ or $\mathcal{H}/\pi_4$.

The following lemma gives a necessary condition for being a normal nilpotent subgroup of an almost Bieberbach group π_3 which is isomorphic to Γ_p.

Lemma 3.1 ([5]). *Let N be a normal nilpotent subgroup of an almost Bieberbach group π_3 and isomorphic to Γ_p. Then N can be represented by one of the following sets of generators*

$$N_1^r = \langle t_1^{d_1},\ t_2^{d_2} t_3^r,\ t_3^{\frac{2nd_1d_2}{p}} \rangle, \qquad N_2^r = \langle t_1^{d_1} t_3^{\frac{2nd_1d_2}{2p}},\ t_2^{d_2} t_3^r,\ t_3^{\frac{2nd_1d_2}{p}} \rangle,$$

$$N_3^\ell = \langle t_1^{d_1} t_2^{\frac{d_2}{2}} t_3^\ell,\ t_2^{d_2} t_3^s,\ t_3^{\frac{2nd_1d_2}{p}} \rangle,$$

where $2d_1$ is a divisor of p, $s = 2\ell$ if $p = 4kd_1$, or $s = 2\ell + \frac{2nd_1d_2}{2p}$ if $p = 2(2k-1)d_1$ for $k \in \mathbb{N}$.

The following proposition is a working criterion for affine conjugacy among normal nilpotent subgroups of π_3.

Proposition 3.2 ([5]). *Let N_1^r, N_2^r, and N_3^ℓ be a normal nilpotent subgroup of π_3 in lemma 3.1 and isomorphic to Γ_p. Then we have the following:*

(1) $N_1^r \sim N_2^{r'}$ *if and only if* $d_2 = p$, $r \equiv r' (mod\ \ d_2)$.
(2) $N_1^r \nsim N_3^\ell$, $N_2^r \nsim N_3^\ell$.
(3) $N_1^r \sim N_1^{r'}$ *if and only if* $r \equiv r' (mod\ \ d_2)$.
(4) $N_2^r \sim N_2^{r'}$ *if and only if* $r \equiv r' (mod\ \ d_2)$.
(5) $N_3^\ell \sim N_3^{\ell'}$ *if and only if* $2\ell \equiv 2\ell' (mod\ \ d_2)$.

Now by using Lemma 3.1 and Proposition 3.2, we can obtain the following result. Note that we deal only with $n = 1$ and $p = 2$ in this paper.

Theorem 3.3. *Suppose G is a finite nonabelian group acting freely on $\mathcal{N}_2$ which yields an orbit manifold homeomorphic to $\mathcal{H}/\pi_3$. Then G is isomorphic to the dihedral group D_4.*

Proof. Note that

$$\pi_3 = \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^2,\ [t_3, t_1] = [t_3, t_2] = 1,\ \ \alpha t_3 \alpha^{-1} = t_3^{-1},$$
$$\alpha t_1 \alpha^{-1} = t_1, \alpha t_2 = t_2^{-1} \alpha t_3^{-1},\ \ \alpha^2 = t_1\ \rangle.$$

Let N be a normal nilpotent subgroup of an almost Bieberbach group π_3 and isomorphic to Γ_2. Then by Lemma 3.1, N can be represented by one of the following sets of generators

$$N_1^r = \langle t_1^{d_1},\ t_2^{d_2} t_3^r,\ t_3^{d_1 d_2} \rangle, \qquad N_2^{r'} = \langle t_1^{d_1} t_3^{\frac{d_1 d_2}{2}},\ t_2^{d_2} t_3^{r'},\ t_3^{d_1 d_2} \rangle,$$
$$N_3^\ell = \langle t_1^{d_1} t_2^{\frac{d_2}{2}} t_3^\ell,\ t_2^{d_2} t_3^s,\ t_3^{d_1 d_2} \rangle,$$

where $2d_1$ is a divisor of 2. In N_3^ℓ, $s = 2\ell$ if $2kd_1 = 1$, or $s = 2\ell + \frac{d_1 d_2}{2}$ if $(2k-1)d_1 = 1$ for $k \in \mathbb{N}$.

Since d_1, d_2, $2d_1$ are divisors of 2, the possible pairs of (d_1, d_2) are $(1,1), (1,2)$.

(I) When $d_1 = 1, d_2 = 1$:

Since $\frac{d_1 d_2}{2} = \frac{1}{2} \notin \mathbb{Z}$, $N_2^{r'}$ and N_3^ℓ do not occur. Thus the possible normal nilpotent subgroup is $N_1^r = \langle\, t_1, t_2 t_3^r, t_3\,\rangle = \langle\, t_1, t_2, t_3\,\rangle$. We denote this group by N. Then, since $N = \langle\, t_1, t_2, t_3\,\rangle \supset [\pi_3, \pi_3] = \langle\, t_2^2 t_3, t_3^2\,\rangle$, π_3/N is abelian. It is not hard to see $\pi_3/N \cong \mathbb{Z}_2$.

(II) When $d_1 = 1, d_2 = 2$:

By Proposition 3.2, we can obtain that $N_1^1 \sim N_2^1$, $N_1^0 \sim N_2^0$, and $N_3^0 \sim N_3^1$. So, we have the following three cases.

(i) $N_1^1 = \langle\, t_1, t_2^2 t_3, t_3^2\,\rangle$.

Note that $N_1^1 \supset [\pi_3, \pi_3] = \langle\, t_2^2 t_3,\, t_3^2\,\rangle$. As in the case of $N = \langle\, t_1,\, t_2,\, t_3\,\rangle$, we can easily check that $\pi_3/N_1^1 \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

(ii) $N_1^0 = \langle\, t_1,\, t_2^2,\, t_3^2\,\rangle$.

Clearly, N_1^0 is a normal subgroup of π_3 and since

$$t_3 N_1^0 = t_3\alpha^2 N_1^0 = t_2 t_2^{-1} t_3 \alpha^2 N_1^0 = t_2 t_2^{-1} \alpha t_3^{-1} \alpha N_1^0 = t_2 \alpha t_2 \alpha N_1^0 = (t_2 \alpha N_1^0)^2,$$
$$t_2 N_1^0 = t_2 \alpha^2 N_1^0 = (t_2 \alpha N_1^0)(\alpha N_1^0), \quad t_1 N_1^0 = \alpha^2 N_1^0,$$

we have

$$\pi_3/N_1^0 = \langle\, t_1,\, t_2,\, t_3,\, \alpha\rangle/\langle\, t_1,\, t_2^2,\, t_3^2\,\rangle = \langle\, t_2\alpha,\, \alpha\rangle/\langle\, t_1,\, t_2^2,\, t_3^2\,\rangle.$$

Also, since

$$\begin{aligned}
&(t_2\alpha N_1^0)^2 = t_2\alpha t_2 \alpha N_1^0 = t_2 t_2^{-1} \alpha t_3^{-1} \alpha N_1^0 = t_3 N_1^0,\\
&(t_2 \alpha N_1^0)^4 = (\alpha N_1^0)^2 = 1,\\
&(\alpha N_1^0)(t_2 \alpha N_1^0) = t_2^{-1} \alpha t_3^{-1} \alpha N_1^0 = t_2^{-1} t_3 \alpha^2 N_1^0 = t_2 t_3 \alpha^2 N_1^0 = t_3 t_2 \alpha^2 N_1^0\\
&\qquad = (t_2\alpha)^2 t_2 \alpha^2 N_1^0 = (t_2\alpha)^3 N_1^0 \alpha N_1^0 = (t_2 \alpha)^{-1} N_1^0 (\alpha N_1^0),
\end{aligned}$$

we obtain that

$$\begin{aligned}
\pi_3/N_1^0 = \langle t_2\alpha N_1^0, \alpha N_1^0 \mid\ & (t_2 \alpha N_1^0)^4 = (\alpha N_1^0)^2 = 1,\\
& (\alpha N_1^0)(t_2\alpha N_1^0) = (t_2\alpha)^{-1} N_1^0 (\alpha N_1^0)\rangle.
\end{aligned}$$

This group is isomorphic to the dihedral group D_4.

(iii) $N_3^0 = \langle\, t_1 t_2,\, t_2^2 t_3,\, t_3^2\,\rangle$.

It is easy to see that N_3^0 is a normal subgroup of π_3 and π_3/N_3^0 is abelian. It needs some calculations to show that $\pi_3/N_3^0 \cong \mathbb{Z}_8$. □

Next we shall deal with the case of π_4. The following lemma gives a necessary condition for being a normal nilpotent subgroup of an almost Bieberbach group π_4 which is isomorphic to Γ_p.

LEMMA 3.4 ([5]). *Let N be a normal nilpotent subgroup of an almost Bieberbach group π_4 and isomorphic to Γ_p. Then N can be represented by one of the following sets of generators: for $s, w \in \mathbb{N}$,*

(A) $p = 4sd_1, \quad p = 2wd_2$:

$$\begin{aligned}
&N_{(1,1)} = \langle t_1^{d_1},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}}\rangle,\ N_{(1,2)} = \langle t_1^{d_1},\ t_2^{d_2} t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}}\rangle,\\
&N_{(2,1)} = \langle t_1^{d_1} t_3^{\frac{4nd_1d_2}{2p}},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}}\rangle,\\
&N_{(2,2)} = \langle t_1^{d_1} t_3^{\frac{4nd_1d_2}{2p}},\ t_2^{d_2} t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}}\rangle,
\end{aligned}$$

$$N_{(3,1)} = \langle t_1^{d_1} t_2^{\frac{d_2}{2}},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}} \rangle,\ N_{(3,3)} = \langle t_1^{d_1} t_2^{\frac{d_2}{2}} t_3^{\frac{4nd_1d_2}{2p}},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}} \rangle.$$

(B) $p = 2(2s-1)d_1, \quad p = 2wd_2$:

$$N_{(1,1)} = \langle t_1^{d_1},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}} \rangle,\ N_{(1,2)} = \langle t_1^{d_1},\ t_2^{d_2} t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}} \rangle,$$
$$N_{(2,1)} = \langle t_1^{d_1} t_3^{\frac{4nd_1d_2}{2p}} t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}} \rangle,$$

$$N_{(2,2)} = \langle t_1^{d_1} t_3^{\frac{4nd_1d_2}{2p}},\ t_2^{d_2} t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}} \rangle,$$
$$N_{(4,1)} = \langle t_1^{d_1} t_2^{\frac{d_2}{2}} t_3^{\frac{12nd_1d_2}{4p}},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}} \rangle,$$
$$N_{(4,3)} = \langle t_1^{d_1} t_2^{\frac{d_2}{2}} t_3^{\frac{4nd_1d_2}{4p}},\ t_2^{d_2},\ t_3^{\frac{4nd_1d_2}{p}} \rangle.$$

(C) $p = 4sd_1, \quad p = (2w-1)d_2$:

$$N_{(3,2)} = \langle t_1^{d_1} t_2^{\frac{d_2}{2}} t_3^{\frac{4nd_1d_2}{4p}},\ t_2^{d_2} t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}} \rangle,$$
$$N_{(3,4)} = \langle t_1^{d_1} t_2^{\frac{d_2}{2}} t_3^{\frac{12nd_1d_2}{4p}},\ t_2^{d_2} t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}} \rangle.$$

(D) $p = 2(2s-1)d_1, \quad p = (2w-1)d_2$:

$$N_{(4,2)} = \langle t_1^{d_1} t_2^{\frac{d_2}{2}},\ t_2^{d_2} t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}} \rangle,$$
$$N_{(4,4)} = \langle t_1^{d_1} t_2^{\frac{d_2}{2}} t_3^{\frac{4nd_1d_2}{2p}},\ t_2^{d_2} t_3^{\frac{4nd_1d_2}{2p}},\ t_3^{\frac{4nd_1d_2}{p}} \rangle.$$

PROPOSITION 3.5 ([5]). *Let $N_{(i,j)}(i, j = 1, 2, 3, 4)$ be a normal nilpotent subgroup of π_4 in lemma 3.4 and isomorphic to Γ_p. Then we have the following:*

(1) $N_{(1,2)} \sim N_{(2,1)}$ *if and only if* $d_1 = d_2$.
$N_{(1,1)} \nsim N_{(1,2)}$, $N_{(1,1)} \nsim N_{(2,1)}$, $N_{(1,1)} \nsim N_{(2,2)}, N_{(1,2)} \nsim N_{(2,2)}$, $N_{(2,1)} \nsim N_{(2,2)}$.
(2) $N_{(3,1)} \nsim N_{(3,3)}$, $N_{(4,1)} \nsim N_{(4,3)}$.
(3) $N_{(3,2)} \sim N_{(3,4)}$ *if and only if* $d_2 = p$.
$N_{(4,2)} \sim N_{(4,4)}$ *if and only if* $d_2 = p$ *or* $2d_1 = p$.
(4) $N_{(1,k)} \nsim N_{(3,j)}$, $N_{(1,k)} \nsim N_{(4,j)}$, $N_{(2,k)} \nsim N_{(3,j)}$, $N_{(2,k)} \nsim N_{(4,j)} (k = 1, 2)$.

By using Lemma 3.4 and Proposition 3.5, we prove that there exist three kinds of nonabelian free actions in π_4.

THEOREM 3.6. *Suppose G is a finite nonabelian group acting freely on $\mathcal{N}_2$ which yields an orbit manifold homeomorphic to $\mathcal{H}/\pi_4$. Then G is isomorphic to D_4(the dihedral group), Q_8(the quaternion group), or $C_8.C_4$(the 1^{st} non-split extension by C_8 of C_4 acting via $C_4/C_2 = C_2$).*

Proof. Note that

$$\begin{aligned}\pi_4 = \langle\, t_1, t_2, t_3, \alpha,\ \beta \mid\ & [t_2, t_1] = t_3^4, [t_3, t_1] = [t_3, t_2] = [\alpha, t_3] = 1,\\ & \beta t_3 \beta^{-1} = t_3^{-1}, \alpha t_1 = t_1^{-1}\alpha t_3^2, \alpha t_2 = t_2^{-1}\alpha t_3^{-2},\\ & \alpha^2 = t_3, \beta^2 = t_1, \beta t_1 \beta^{-1} = t_1, \beta t_2 = t_2^{-1}\beta t_3^{-2},\\ & \alpha\beta = t_1^{-1} t_2^{-1}\beta\alpha t_3^{-3}\rangle.\end{aligned}$$

Let N be a normal nilpotent subgroup of an almost Bieberbach group π_4 and isomorphic to Γ_2. Note that d_1, d_2 are divisors of 2. So, by Lemma 3.4, the cases (A) and (C) do not occur. Therefore we only have to deal with the cases (B) and (D).

(I) $p = 2(2s-1)d_1, \quad p = 2wd_2$:

In this case, we must have $d_1 = d_2 = 1$ and then $\frac{d_2}{2} = \frac{1}{2} \notin \mathbb{Z}$. So, there exist four possible normal nilpotent subgroups

$$N_{(1,1)} = \langle\, t_1,\ t_2,\ t_3^2\,\rangle, \qquad N_{(1,2)} = \langle\, t_1,\ t_2 t_3,\ t_3^2\,\rangle,$$

$$N_{(2,1)} = \langle\, t_1 t_3,\ t_2,\ t_3^2\,\rangle, \qquad N_{(2,2)} = \langle\, t_1 t_3,\ t_2 t_3,\ t_3^2\,\rangle.$$

Since $d_1 = d_2 = 1$, by Proposition 3.5, it follows that $N_{(1,2)}$ is affinely conjugate to $N_{(2,1)}$.

(i) $N_{(1,1)} = \langle\, t_1, t_2, t_3^2\,\rangle$.

The following relations

$$\begin{aligned}&\alpha t_1 \alpha^{-1} = t_1^{-1}\alpha t_3^2 \alpha^{-1} = t_1^{-1} t_3^2 \in N_{(1,1)},\ \beta t_1 \beta^{-1} = t_1 \in N_{(1,1)},\\ &\alpha t_2 \alpha^{-1} = t_2^{-1}\alpha t_3^{-2}\alpha^{-1} = t_2^{-1} t_3^{-2} \in N_{(1,1)},\\ &\beta t_2 \beta^{-1} = t_2^{-1}\beta t_3^{-2}\beta^{-1} = t_2^{-1} t_3^2 \in N_{(1,1)},\\ &\beta t_3^2 \beta^{-1} = t_3^{-2} \in N_{(1,1)},\ \alpha^2 = t_3,\ \beta^2 = t_1\end{aligned}$$

show that $N_{(1,1)}$ is a normal subgroup of π_4 and

$$\pi_4/N_{(1,1)} = \langle\, t_1, t_2, t_3, \alpha, \beta\,\rangle / \langle\, t_1, t_2, t_3^2\,\rangle = \langle\, \alpha N_{(1,1)}, \beta N_{(1,1)}\,\rangle.$$

Since

$$\begin{aligned}&(\alpha N_{(1,1)})^4 = t_3^2 N_{(1,1)} = 1,\ (\beta N_{(1,1)})^2 = t_1 N_{(1,1)} = 1,\\ &\beta\alpha N_{(1,1)} = t_2 t_1 \alpha\beta t_3^3 N_{(1,1)} = \alpha\beta t_3 N_{(1,1)} = \alpha t_3^{-1}\beta N_{(1,1)} = \alpha^{-1}\beta N_{(1,1)},\end{aligned}$$

we have

$$\pi_4/N_{(1,1)} = \langle\, \alpha N_{(1,1)}, \beta N_{(1,1)} \,|\, (\alpha N_{(1,1)})^4 = (\beta N_{(1,1)})^2 = 1, \\ (\beta N_{(1,1)})(\alpha N_{(1,1)}) \;=\; (\alpha N_{(1,1)})^{-1}(\beta N_{(1,1)}) \,\rangle.$$

This group is isomorphic to the dihedral group D_4.

(ii) $N_{(1,2)} = \langle\, t_1, t_2t_3, t_3^2 \,\rangle$.

Since $N_{(1,2)} \supset [\pi_4, \pi_4] = \langle t_1^2, t_2^2, t_3^2, t_1t_2t_3\rangle$, it is easy to check $\pi_4/N_{(1,2)} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, which is the same as the result of [1, Theorem 3.5].

(iii) $N_{(2,2)} = \langle\, t_1t_3, t_2t_3, t_3^2 \,\rangle$.

The following relations

$$\begin{aligned}
&\alpha t_1t_3\alpha^{-1} = t_1^{-1}\alpha t_3^2t_3\alpha^{-1} = t_1^{-1}t_3^3 = (t_1t_3)^{-1}t_3^4 \in N_{(2,2)},\\
&\beta t_1t_3\beta^{-1} = t_1\beta\beta^{-1}t_3^{-1} = t_1t_3^{-1} = (t_1t_3)t_3^{-2} \in N_{(2,2)},\\
&\alpha t_2t_3\alpha^{-1} = t_2^{-1}\alpha t_3^{-2}t_3\alpha^{-1} = (t_2t_3)^{-1} \in N_{(2,2)},\\
&\beta t_2t_3\beta^{-1} = t_2^{-1}\beta t_3^{-2}t_3\beta^{-1} = t_2^{-1}\beta t_3^{-1}\beta^{-1} = t_2^{-1}t_3 = (t_2t_3)^{-1}t_3^2 \in N_{(2,2)}
\end{aligned}$$

show that $N_{(2,2)}$ is a normal subgroup of π_4. Also, from the following relations,

$$\begin{aligned}
&t_2N_{(2,2)} = t_2t_3^2N_{(2,2)} = t_3N_{(2,2)} = \alpha^2N_{(2,2)},\\
&(\beta N_{(2,2)})^2 = t_1t_3t_3^{-1}N_{(2,2)} = t_3N_{(2,2)} = (\alpha N_{(2,2)})^2,\\
&(\alpha N_{(2,2)})^4 = t_3^2N_{(2,2)} = 1,\\
&\beta\alpha N_{(2,2)} = t_2t_1\alpha\beta t_3^3N_{(2,2)} = \alpha\beta t_3N_{(2,2)} = \alpha t_3^{-1}\beta N_{(2,2)} = \alpha^{-1}\beta N_{(2,2)},
\end{aligned}$$

it follows that

$$\pi_4/N_{(2,2)} = \langle\, \alpha N_{(2,2)}, \beta N_{(2,2)} \,|\, (\alpha N_{(2,2)})^4 = 1, (\alpha N_{(2,2)})^2 = (\beta N_{(2,2)})^2, \\ (\beta N_{(2,2)})(\alpha N_{(2,2)}) \;=\; (\alpha N_{(2,2)})^{-1}(\beta N_{(2,2)}) \,\rangle.$$

This group is isomorphic to the quaternion group Q_8.

(II) $p = 2(2s-1)d_1, \quad p = (2w-1)d_2$:

In this case, we have $d_1 = 1$ and $d_2 = 2$. So, we have two possible normal nilpotent subgroups $N_{(4,2)} = \langle\, t_1t_2,\, t_2^2t_3^2,\, t_3^4 \,\rangle$, $N_{(4,4)} = \langle\, t_1t_2t_3^2,\, t_2^2t_3^2,\, t_3^4 \,\rangle$. By Proposition 3.5, we know that $N_{(4,4)}$ is affinely conjugate to $N_{(4,2)}$. Therefore there exists only one normal nilpotent subgroup $N_{(4,2)}$. Normality of $N_{(4,2)}$ can be easily checked as in the case of (iii). Note that

$$
\begin{aligned}
&(\alpha N_{(4,2)})^8 = t_3^4 N_{(4,2)} = 1,\ (\beta N_{(4,2)})^2 = t_1 N_{(4,2)},\\
&(\beta N_{(4,2)})^3 = \beta t_1 N_{(4,2)} = \beta t_1 t_2 t_2^{-1} N_{(4,2)} = \beta t_2^{-1} N_{(4,2)} = \beta t_2 t_3^2 N_{(4,2)},\\
&(\beta N_{(4,2)})^4 = \beta^2 t_2 t_3^2 N_{(4,2)} = t_1 t_2 t_3^2 N_{(4,2)} = t_3^2 N_{(4,2)} = \alpha^4 N_{(4,2)} = (\alpha N_{(4,2)})^4,\\
&\quad t_2 N_{(4,2)} = t_1^{-1} t_1 t_2 N_{(4,2)} = t_1^{-1} N_{(4,2)} = \beta^{-2} N_{(4,2)}.
\end{aligned}
$$

Let $a = \alpha\beta^2 N_{(4,2)}$ and $b = \beta N_{(4,2)}$. Then we have

$$\pi_4/N_{(4,2)} = \langle \alpha N_{(4,2)}, \beta N_{(4,2)} \rangle = \langle\, a, b\,\rangle.$$

From the following relation,

$$
\begin{aligned}
\beta\alpha N_{(4,2)} &= t_2 t_1 \alpha\beta t_3^3 N_{(4,2)} = t_1 t_2 t_3^4 \alpha\beta t_3^3 N_{(4,2)} = \alpha\beta t_3^3 N_{(4,2)}\\
&= \alpha\beta t_3^{-1} N_{(4,2)} = \alpha t_3 \beta N_{(4,2)} = \alpha^3 \beta N_{(4,2)},
\end{aligned}
$$

we can obtain that

$$
\begin{aligned}
(\alpha\beta^2 N_{(4,2)})(\alpha^3\beta^2 N_{(4,2)}) &= \alpha\beta^2\alpha^3\beta^2 N_{(4,2)} = \alpha^{28}\beta^4 N_{(4,2)}\\
&= \alpha^4\beta^4 N_{(4,2)} = \alpha^8 N_{(4,2)} = 1.
\end{aligned}
$$

This implies $a^{-1} = \alpha^3\beta^2 N_{(4,2)}$. Hence

$$bab^{-1} = \beta(\alpha\beta^2)\beta^7 N_{(4,2)} = \beta\alpha\beta N_{(4,2)} = \alpha^3\beta^2 N_{(4,2)} = a^{-1}$$

and

$$
\begin{aligned}
&a^2 = (\alpha\beta^2)(\alpha\beta^2) N_{(4,2)} = \alpha^{10}\beta^4 N_{(4,2)} = \alpha^2\beta^4 N_{(4,2)},\\
&a^4 = (\alpha^2\beta^4)(\alpha^2\beta^4) N_{(4,2)} = \alpha^{164}\beta^8 N_{(4,2)} = \alpha^4 N_{(4,2)} = \beta^4 N_{(4,2)} = b^4,\\
&a^8 = 1.
\end{aligned}
$$

Therefore $\pi_4/N_{(4,2)} = \langle\, t_1, t_2, t_3, \alpha, \beta\,\rangle / \langle\, t_1 t_2, t_1^2 t_2^2, t_3^4\,\rangle$ is isomorphic to the group

$$C_8.C_4 = \langle a, b \,|\, a^8 = 1,\ a^4 = b^4,\ bab^{-1} = a^{-1}\rangle,$$

that is, the 1^{st} non-split extension by C_8 of C_4 acting via $C_4/C_2 = C_2$. □

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